for small k-values are given by

$$\delta = -k\,a - \frac{2\,k^2 Z \pi}{3\,A^2 B} - \frac{8\,Z}{3\,A^2 B}\,k^3 \ln|k| + O(k^3)\,. \quad (14)$$

Since a is known, δ can be calculated. Using Eq. (14) for the phase shifts we can calculate the total cross-sections σ for the elastic scattering of low energy electrons within the framework of the Thomas-Fermi theory by means of the well known equation.

$$\sigma = (4\,\pi/k^2)\,\sin^2\delta\,. \quad (15)$$

In Table 2 we have collected results for σ for certain elements belonging to the same groups in the periodic table.

Table 2 contains the numerical values of the total cross section for several elements. The results show that σ is not a monotonic function of k for small k-values but neither a complicated function of k. σ contains some maxima and minima which means it ahs an oscillatory character of low energy scattering of electrons by atoms in the Thomas-Fermi theory.

I wish to thank Professor R. G. Breene, Jr., for inviting me to the West Virginia University as a visiting Professor and for his assistance in preparing this paper.

Kinetic Theory for a Dilute Gas of Particles with “Spin”

III. The Influence of Collinear Static and Oscillating Magnetic Fields on the Viscosity

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(Z. Naturforsch. **26 a**, 1057—1071 [1971]; received 3 February 1971)

The Senftleben-Beenakker effect of the viscosity of dilute polyatomic gases is investigated theoretically for the case where an alternating magnetic field parallel to the usual static field is present. The starting point is a set of transport-relaxation equations obtained from the kinetic equation for the one-particle distribution by application of the moment method. The transport-relaxation equations are solved for the viscosity problem and the relevant viscosity coefficients averaged over many periods of the oscillating field are calculated as functions of the frequency of the alternating field and of the magnitudes of both magnetic fields. The importance of the obtained results for the dynamic behaviour of the thermomagnetic gas torque (Scott effect) is discussed.

The transport properties, in particular heat conductivity and viscosity of dilute polyatomic gases, are influenced by an external magnetic field[1] (Senftleben-Beenakker effects). It is the purpose of this paper to present a theoretical investigation of the effect of two collinear static and alternating magnetic fields on the viscosity of a dilute gas of polyatomic molecules. In view of the close connection between the Senftleben-Beenakker effects for the heat conductivity and viscosity and the “thermomagnetic torque effect” observed for rarefied polyatomic gases[2] (Scott effect), such an analysis is also of importance for recent Scott effect measurements[3] involving collinear static and alternating magnetic fields. A theoretical analysis of this “dynamic behaviour” of the thermomagnetic torque has already been presented by the authors[4].

The Senftleben-Beenakker effects have been studied extensively (both experimentally and theoretically) during the last decade; for an excellent review of the literature see the article by Beenakker and McCourt[5]. Qualitatively, the influence of a magnetic field on the transport properties of polyatomic gases can be understood as follows. In the transport situation, collisions between molecules possessing a nonspherical interaction give rise to

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an alignment of the internal rotational angular momenta or to a correlation between rotational and linear momenta of the molecules. This alignment, in turn, affects the value of the transport coefficients. Hence if the alignment is (partially) destroyed by the precessional motion of the rotational angular momentum about the direction of an applied magnetic field, the transport coefficients are altered.

For a theoretical treatment within the framework of kinetic theory one has to start from a kinetic equation or generalized Boltzmann equation for the one-particle distribution function characterizing the nonequilibrium state of the gas. The appropriate quantum mechanical kinetic equation for gases consisting of particles with internal rotational degrees of freedom has first been derived by WALDMANN [6] and later independently by SNIDER [7, 8]. A solution procedure for this kinetic equation which is akin to the Chapman-Enskog method has been used by MCCOURT and SNIDER [9] to treat the Senftleben-Beenakker effect on heat conductivity and viscosity. Previously, KAGAN and MAKSIMOW [10] had already treated the influence of a magnetic field on the heat conductivity by the same method but they started from a (simplified) classical kinetic equation for a gas of rotating molecules; an extension of their work to the influence of a magnetic field on the viscosity has been given by KNAAP and BEENAKKER [11]. For the further development of the application of the Chapman-Enskog method to the study of the Senftleben-Beenakker effects Ref. [5] should be consulted.

A different solution procedure for the quantum mechanical kinetic equation – the moment method – has been developed by WALDMANN and coworkers [12, 13]. Within the moment method – contrary to the Chapman-Enskog method – one is not restricted to "normal solutions" of the kinetic equation. This point is of crucial importance for the study of time-dependent phenomena; e. g. in the presence of alternating external fields. This paper can be considered as part III of our series on "the kinetic theory for a gas of paricles with spin". Although in parts I and II (see Ref. [13]) we have mainly been concerned with the "single level case" (molecules with one rotational level; e. g. o-H_2 at low temperatures), in this paper we consider the more general case of rotating molecules which may occupy different rotational energy levels. The influence of an oscillating field and of perpendicular static and oscillating magnetic fields on transport properties has previously been investigated by BORMAN et al. [14] The influence of collinear static and alternating magnetic fields on the viscosity has recently also been studied theoretically by MCCOURT and MORAAL [14a].

In this paper, first the coupled transport-relaxation equations for the friction pressure tensor and the tensor polarization of the rotational angular momentum are stated (§ 1). Section 2 presents the solution of the viscosity problem in the absence of a magnetic field. Section 3 is devoted to the definition of the viscosity coefficients characterizing the viscosity tensor in the presence of a magnetic field. Then, in Section 4, these viscosity coefficients averaged over many periods of the oscillating field are calculated for the case where two collinear magnetic fields, a static and an oscillating one, are present. Some special cases of the obtained formulae are discussed in Section 5. Finally, the dependence of the viscosity coefficients on the frequency of the alternating field is studied in more detail and the relevance of these results for the dynamic behaviour of the thermomagnetic torque effect is indicated (§ 6). The Appendix deals with the projection

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tensors used for the solution of the viscosity problem and with the physical meaning of the various viscosity coefficients.

§ 1. Transport-Relaxation Equations

a) Choice of Macroscopic Variables

The Senftleben-Beenakker effect is intimately linked with the fact that transport processes in polyatomic gases give rise to an alignment of the molecular rotational angular momentum. Hence a quantity characterizing this alignment has to be included in the set of “macroscopic variables” needed to specify the nonequilibrium state of the gas. The usual macroscopic variables required for a treatment of the viscosity problem are the flow velocity $\boldsymbol{v}$, the scalar pressure p and the (traceless) friction pressure tensor $\mathbf{p}$. The alignment which is the most important one in the presence of a viscous flow in a gas of linear molecules is described by a kind of a (2nd rank) tensor polarization

$$\langle R \, \overline{\boldsymbol{J}\boldsymbol{J}} \rangle \, . \tag{1.1}$$

Here $\boldsymbol{J}$ is the rotational angular momentum operator (in units of $\hbar$) and R is a scalar function which may depend on the magnitude of $\boldsymbol{J}$ and of the linear momentum $\boldsymbol{p}$ of a molecule. The symbol “$\overline{\cdots}$” designates the anisotropic, i. e. symmetric traceless (irreducible) part of a tensor. The bracket $\langle \ldots \rangle$ refers to an average evaluated with the one-particle distribution operator $f(t, \boldsymbol{x}, \boldsymbol{p}, \boldsymbol{J})$ of the gas. That is to say

$$\langle \Psi \rangle = n^{-1} \operatorname{Tr} \int \mathrm{d}^3 p \; \Psi(\boldsymbol{p}, \boldsymbol{J}) \, f(t, \boldsymbol{x}, \boldsymbol{p}, \boldsymbol{J}) \tag{1.2}$$

gives the local instantaneous mean value of some operator $\Psi = \Psi(\boldsymbol{p}, \boldsymbol{J})$. In Eq. (1.2), “Tr” denotes the trace over magnetic quantum numbers and summation over rotational quantum numbers, and

$$n(t, \boldsymbol{x}) = \operatorname{Tr} \int \mathrm{d}^3 p \, f(t, \boldsymbol{x}, \boldsymbol{p}, \boldsymbol{J}) \tag{1.3}$$

is the number density.

It has been discussed earlier [15] that it is not possible to determine from measurements of the Senftleben-Beenakker effect on the viscosity alone how R should be chosen for a “single-term description” of the effect. However, the type of alignment occurring in connection with flow birefringence [16, 17] and depolarized Rayleigh light scattering [15, 17, 18] in gases of linear molecules is of the form (1.1) with $R = (J^2 - \tfrac{3}{4})^{-1}$. There are strong indications – based on the comparison of the experimentally determined width of the depolarized Rayleigh line [19] with data extracted from measurements of the Senftleben-Beenakker effect of the viscosity [5, 20, 21] – that the tensor polarization associated with the influence of a magnetic field on the viscosity is also of this form. Thus as the macroscopic variable characterizing the alignment of the molecular rotational angular momentum the tensor polarization

$$\mathbf{a} = \langle \boldsymbol{\Phi} \rangle \tag{1.4}$$

with

$$\boldsymbol{\Phi} = \sqrt{\frac{15}{2}} \left(\left\langle \frac{J^2}{J^2 - \frac{3}{4}} \right\rangle_0 \right)^{-1/2} (J^2 - \tfrac{3}{4})^{-1} \, \overline{\boldsymbol{J}\boldsymbol{J}} \tag{1.5}$$

is chosen. The bracket $\langle \ldots \rangle_0$ refers to an average taken with an equilibrium distribution f_0. The quantity $\boldsymbol{\Phi}$ has been normalized such that

$$\langle \boldsymbol{\Phi} \boldsymbol{\Phi} \rangle_0 = \boldsymbol{\Delta} \, . \tag{1.6}$$

In Cartesian component notation the isotropic tensor $\boldsymbol{\Delta}$ (of rank 4) is given by

$$\Delta_{\mu\nu, \mu'\nu'} = \tfrac{1}{2} \left(\delta_{\mu\mu'} \, \delta_{\nu\nu'} + \delta_{\mu\nu'} \, \delta_{\nu\mu'} \right) - \tfrac{1}{3} \, \delta_{\mu\nu} \, \delta_{\mu'\nu'} \, . \tag{1.7}$$

b) The Transport-Relaxation Equations

By application of the moment method [12, 13] to the kinetic equation [6–8] for the nonequilibrium distribution operator f the following set of transport-relaxation equations can be derived for $\mathbf{p}$ and $\mathbf{a}$:

$$\frac{\partial \mathbf{p}}{\partial t} + 2 \, p_0 \, \overline{\nabla \boldsymbol{v}} + \omega_\eta \, \mathbf{p} + \sqrt{2} \, p_0 \, \omega_{\eta \mathrm{T}} \, \mathbf{a} = 0 \, , \tag{1.8}$$

$$\frac{\partial \mathbf{a}}{\partial t} + \gamma \, H \, \mathcal{H} : \mathbf{a} + \frac{1}{\sqrt{2} \, p_0} \, \omega_{\eta \mathrm{T}} \, \mathbf{p} + \omega_{\mathrm{T}} \, \mathbf{a} = 0 \, . \tag{1.9}$$

Here p_0 is the (constant) equilibrium pressure. The “relaxation coefficients” ω_η, $\omega_{\eta \mathrm{T}}$ and ω_{T} can be

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expressed in terms of "collision brackets". These brackets are obtained from the linearized collision operator $\omega(\ldots)$ of the quantum mechanical kinetic equation. In particular, one has

$$\omega_\eta = \frac{2}{5} \langle \overline{V_\mu V_\nu}\, \omega(\overline{V_\mu V_\nu}) \rangle_0 , \qquad (1.10)$$

$$\omega_{\eta\mathrm{T}} = \frac{\sqrt{2}}{5} \langle \overline{V_\mu V_\nu}\, \omega(\Phi_{\mu\nu}) \rangle_0 = \frac{\sqrt{2}}{5} \langle \Phi_{\mu\nu}\, \omega(\overline{V_\mu V_\nu}) \rangle_0 , \qquad (1.11)$$

$$\omega_\mathrm{T} = \frac{1}{5} \langle \Phi_{\mu\nu}\, \omega(\Phi_{\mu\nu}) \rangle_0 , \qquad (1.12)$$

where $\mathbf{V}$ is the linear momentum of a particle in units of $\sqrt{2\, m\, k\, T_0}$. Here m is the mass of a molecule and T_0 is the equilibrium temperature of the gas. In Ref. [13] the relaxation coefficients have been denoted by $\omega^{(11)}_{+2}$, $\omega^{(12)}_{+2}$, and $\omega^{(22)}_{+2}$. The relaxation coefficient ω_T for the tensor polarization determines the pressure broadening of the depolarized Rayleigh line[17–19]. The coefficient $\omega_{\eta\mathrm{T}}$ which characterizes the strength of the collision-induced coupling between the friction pressure tensor and the tensor polarization is also of crucial importance for flow birefringence[17].

Effective cross sections pertaining to the relaxation coefficients can be introduced by

$$\omega_{\cdot\cdot} = n_0 \sqrt{\frac{8\, k\, T_0}{\pi\, m_{12}}}\, \sigma_{\cdot\cdot} \qquad (1.13)$$

where ".." stands for "η", "$\eta\, T$" and "T". The "cross section" $\sigma_{\eta\mathrm{T}}$ may be positive or negative, the others are positive. In Eq. (1.13) n_0 is the equilibrium number density and $m_{12} = \frac{1}{2} m$ is the reduced mass of two molecules.

The 2nd term in Eq. (1.9) describes the precessional motion of the tensor polarization of a gas of molecules with a (constant) gyromagnetic ratio γ in the presence of a magnetic field $\boldsymbol{H} = H\, \boldsymbol{h}$ ($\boldsymbol{h}$ is a unit vector). In Cartesian component notation the 4-th rank tensor $\mathcal{H}$ is given by

$$\mathcal{H}_{\mu\nu,\,\mu'\nu'} = h_\lambda (\varepsilon_{\mu\lambda\mu'}\, \delta_{\nu\nu'} + \varepsilon_{\nu\lambda\nu'}\, \delta_{\mu\mu'}) \qquad (1.14)$$

where $\varepsilon_{\mu\nu\lambda}$ is the isotropic tensor of rank 3 with $\varepsilon_{123} = 1$. The product $\mathcal{H} : \mathbf{a}$ is

$$(\mathcal{H} : \mathbf{a})_{\mu\nu} = \varepsilon_{\mu\lambda\varrho}\, h_\lambda\, a_{\varrho\nu} + \varepsilon_{\nu\lambda\varrho}\, h_\lambda\, a_{\mu\varrho} , \qquad (1.15)$$

it essentially describes the infinitesimal rotation of a 2-nd rank tensor about a direction specified by $\boldsymbol{h}$. Some of its further properties are discussed in the Appendix.

Here, as in all previous work on the Senftleben-Beenakker effect it is assumed that the collisions are not affected by the magnetic field.

c) Alternative Form of the Transport-Relaxation Equations

It proves convenient for the treatment of the viscosity problem to introduce the "friction pressure tensor" $\mathbf{p}_\mathrm{T}$ associated with the tensor polarization by

$$\mathbf{p}_\mathrm{T} = -\sqrt{2}\, p_0\, \omega_{\eta\mathrm{T}} / \omega_\eta\, \mathbf{a} . \qquad (1.16)$$

Then Eq. (1.8) can be written in the form

$$\omega_\eta^{-1} \frac{\partial \mathbf{p}}{\partial t} + \mathbf{p} = -2 \frac{p_0}{\omega_\eta} \overline{\nabla \boldsymbol{v}} + \mathbf{p}_\mathrm{T} , \qquad (1.17)$$

and Eq. (1.9) is equivalent to

$$\frac{\partial \mathbf{p}_\mathrm{T}}{\partial t} + \gamma\, H\, \mathcal{H} : \mathbf{p}_\mathrm{T} + \omega_\mathrm{T}\, \mathbf{p}_\mathrm{T} = \omega_\mathrm{T}\, A_{\eta\mathrm{T}}\, \mathbf{p} \qquad (1.18)$$

where $A_{\eta\mathrm{T}}$ is given by

$$A_{\eta\mathrm{T}} = \omega_{\eta\mathrm{T}}^2 / \omega_\eta\, \omega_\mathrm{T} . \qquad (1.19)$$

§ 2. Field Free Viscosity

Before treating the viscosity in the presence of a constant magnetic field it is instructive to consider the steady state situation where no magnetic field is present.

In steady state the time derivatives of $\mathbf{p}$ and $\mathbf{a}$ are equal to zero in Eqs. (1.17), (1.18). Then, for $H = 0$, Eq. (1.18) yields

$$\mathbf{p}_\mathrm{T} = A_{\eta\mathrm{T}}\, \mathbf{p} , \qquad (2.1)$$

and Eq. (1.17) can be solved for $\mathbf{p}$ to give

$$\mathbf{p} = -2\, \eta\, \overline{\nabla \boldsymbol{v}} \qquad (2.2)$$

with

$$\eta = \eta_\mathrm{iso} (1 - A_{\eta\mathrm{T}})^{-1} . \qquad (2.3)$$

Here

$$\eta_\mathrm{iso} = p_0 / \omega_\eta \qquad (2.4)$$

is the "isotropic viscosity" for the case where no alignment (tensor polarization) would be set up in a viscous flow ($A_{\eta\mathrm{T}} = 0$). The actual viscosity η, however, is larger than η_iso. Now, if the tensor polarization is partially destroyed by the precessional motion of the rotational angular momenta about an applied magnetic field, the viscosity will decrease and approach its "isotropic" value η_iso. Thus the maximal field-induced relative change of the viscosity turns out to be

$$(\Delta\eta/\eta)_{\max} = (\eta - \eta_{\text{iso}})/\eta = A_{\eta\text{T}} . \qquad (2.5)$$

Measurements of the Senftleben-Beenakker effect on the viscosity show that $A_{\eta\text{T}}$ is of the order of 10^{-2} and smaller [5, 20, 21]. Hence terms of 2nd and higher power in $A_{\eta\text{T}}$ can be neglected in the viscosity, and Eq. (2.3) can be replaced by

$$\eta = \eta_{\text{iso}}(1 + A_{\eta\text{T}}) . \qquad (2.6)$$

In the presence of a magnetic field the shear viscosity becomes a 4th rank tensor characterized by 5 viscosity coefficients. Before the viscosity is calculated from Eqs. (1.17) and (1.18) the definition of a set of viscosity coefficients will be given.

§ 3. Viscosity Tensor and the Definition of the Viscosity Coefficients

The choice of the 5 linearly independent viscosity coefficients characterizing the shear viscosity tensor in the presence of a magnetic field is arbitrary to some extent. Here two sets of viscosity coefficients are introduced which have certain advantages for the treatment of the viscosity problem [i. e. for the solution of Eq. (1.17), (1.18)] and which are related to measurable quantities in a rather simple way.

The first set of viscosity coefficients $\eta^{(m)}$ $(\text{m} = 0, \pm 1, \pm 2)$ is defined by [17]

$$\mathbf{p} = -2 \sum_{m=-2}^{+2} \eta^{(m)} \mathcal{P}^{(m)} : \overline{\nabla \boldsymbol{v}} \qquad (3.1)$$

where the $\mathcal{P}^{(m)}$ are 4th rank "projection tensors" with the properties:

$$\mathcal{P}^{(m)} : \mathcal{P}^{(m')} = \mathcal{P}^{(m)} \delta^{mm'} , \qquad (3.2)$$

$$\mathcal{P}^{(m)} : \mathcal{H} = \mathcal{H} : \mathcal{P}^{(m)} = \text{i}m \, \mathcal{P}^{(m)} , \qquad (3.3)$$

$$\sum_{m=-2}^{+2} \mathcal{P}^{(m)} : \mathbf{A} = \mathbf{A} , \qquad (3.4)$$

where $\mathbf{A}$ in any 2nd rank tensor.

For $\mathcal{H}$ see Eq. (1.13). For further properties of the $\mathcal{P}^{(m)}$ and their connection with the rotation of a 2nd rank tensor about a direction specified by the unit vector $\boldsymbol{h}$ see the Appendix. Since $\mathcal{P}^{(m)*} = \mathcal{P}^{(-m)}$ one has

$$\eta^{(m)*} = \eta^{(-m)} . \qquad (3.5)$$

Viscosity coefficients of this type have also been used by MORAAL and MCCOURT [22]. Similar coefficients arise when spherical tensors instead of Cartesian tensors are used, cf. Ref. [23, 24]. The $\eta^{(m)}$ are related to the viscosity coefficients $\eta_1, \ldots, \eta_5$ of DEGROOT and MAZUR [25] by

$$\eta^{(0)} = \eta_1 , \quad \eta^{(1)} = \eta_3 + i\,\eta_5 , \quad \eta^{(2)} = 2\,\eta_2 - \eta_1 - i\,\eta_4 . \qquad (3.6)$$

Without a magnetic field one has $\eta^{(0)} = \eta^{(1)} = \eta^{(2)}$ and the viscosity tensor is isotropic. A set of real "viscosity" coefficients which characterize the deviation of the viscosity tensor from isotropy can be introduced by ("trans" refers to "transverse")

$$\mathbf{p} = -2\,\eta \left\{ 1 + \varepsilon_0 \mathcal{P}^{(0)} + \sum_{m=1}^{2} \left[\varepsilon_m (\mathcal{P}^{(m)} + \mathcal{P}^{(-m)}) + i\,\varepsilon_m^{\text{trans}} (\mathcal{P}^{(m)} - \mathcal{P}^{(-m)}) \right] \right\} : \overline{\nabla \boldsymbol{v}} \qquad (3.7)$$

The scalar η is the viscosity without magnetic field and the coefficients ε_m, $\varepsilon_m^{\text{trans}}$ are related to the complex viscosity coefficients $\eta^{(m)}$ by

$$\varepsilon_m = \eta^{-1}(\text{Re}\,\eta^{(m)} - \eta) ; \quad \varepsilon_m^{\text{trans}} = \eta^{-1}\,\text{Im}\,\eta^{(m)} ; \; m = 0, 1, 2. \qquad (3.8)$$

The relation of the ε_m and $\varepsilon_m^{\text{trans}}$ to measurable quantities is discussed in the Appendix. The possibility of measuring the magnetic field-induced changes of all 5 linearly independent viscosity coefficients has been demonstrated by HULSMAN and BURGMANS [26].

§ 4. Calculation of the Viscosity Coefficients

a) Constant Magnetic Field

The case of a constant magnetic field is considered first. As in the field-free case the time derivatives occurring in Eqs. (1.17, 18) vanish in a steady state situation. Then one has

$$\mathbf{p} = -2\,\eta_{\text{iso}}\,\overline{\nabla \boldsymbol{v}} + \mathbf{p}_{\text{T}} , \qquad (4.1)$$

and

$$\varphi \mathcal{H} : \mathbf{p}_{\text{T}} + \mathbf{p}_{\text{T}} = -2\,\eta_{\text{iso}}\,A_{\eta\text{T}}\,\overline{\nabla \boldsymbol{v}} , \qquad (4.2)$$

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where on the right hand side of Eq. (1.18) $\mathbf{p}$ has been replaced by its "isotropic value". This means that terms of 2nd and higher order in $A_{\eta\mathrm{T}}$ are neglected. The quantity

$$\varphi = \gamma\, H/\omega_\mathrm{T} = \omega_\mathrm{H}/\omega_\mathrm{T} \tag{4.3}$$

is the angle over which the molecular rotational angular momentum precesses during the effective collision time ω_T^{-1}. Since ω_T is proportional to the number density one has, for constant temperature, $\varphi \propto H/p_0$ where p_0 is the equilibrium pressure.

The solution of (4.2) is most easily obtained by splitting $\mathbf{p}_\mathrm{T}$ into "eigentensors" $\mathbf{p}_\mathrm{T}^{(m)}$ of the operator $\mathcal{H}$ according to

$$\mathbf{p}_\mathrm{T} = \sum_{m=-2}^{+2} \mathbf{p}_\mathrm{T}^{(m)} \tag{4.4}$$

with

$$\mathbf{p}_\mathrm{T}^{(m)} = \mathcal{P}^{(m)} : \mathbf{p}_\mathrm{T}\,. \tag{4.5}$$

Then recalling (3.3), Eq. (4.2) yields

$$\mathbf{p}_\mathrm{T} = -2\,\eta_\mathrm{iso}\, A_{\eta\mathrm{T}} \sum_m (1 + i\,m\,\varphi)^{-1}\, \mathcal{P}^{(m)} : \overline{\nabla \boldsymbol{v}}\,. \tag{4.6}$$

If (4.6) is inserted into (4.1) a relation of the form (3.1) is found and the viscosity coefficients $\eta^{(m)}$ are given by

$$\begin{aligned} \eta^{(m)} &= \eta_\mathrm{iso}[1 + A_{\eta\mathrm{T}}(1 + i\,m\,\varphi)^{-1}] \\ &= \eta\ \ [1 - A_{\eta\mathrm{T}} i\,m\,\varphi(1 + i\,m\,\varphi)^{-1}], \\ & \qquad m = 0, \pm 1, \pm 2. \end{aligned} \tag{4.7}$$

The ensuing real coefficients ε_m and $\varepsilon_m^\mathrm{trans}$ are (for $m = 0, 1, 2$)

$$\varepsilon_m \quad = -A_{\eta\mathrm{T}} \frac{m^2\,\varphi^2}{1 + m^2\,\varphi^2}\,, \tag{4.8}$$

$$\varepsilon_m^\mathrm{trans} = -A_{\eta\mathrm{T}} \frac{m\,\varphi}{1 + m^2\,\varphi^2}\,. \tag{4.9}$$

Eqs. (4.8), (4.9) are equivalent to the results first calculated by McCourt and Snider [9] for the Senftleben-Beenakker effect of the viscosity.

b) Collinear Static and Alternating Magnetic Fields

Now a magnetic field of the form

$$H(1 + \beta \sin \omega\, t)\, \mathbf{h} \tag{4.10}$$

is considered. Here H is the magnitude of the constant field. The ratio of the magnitude of the alternating field (with angular frequency ω) to the magnitude of the constant field is denoted by β.

In the presence of an alternating magnetic field both $\mathbf{p}$ and $\mathbf{p}_\mathrm{T}$ are time-dependent quantities. In a streaming experiment or with the Scott pendulum, however, time-averaged quantities are measured. The time average $\bar{a}$ of a quantity $a(t)$ is defined by

$$\bar{a} = \lim_{T\to\infty} \frac{1}{T} \int_0^T a(t)\, \mathrm{d}t\,. \tag{4.11}$$

For a steady flow (i. e. $\overline{\nabla \boldsymbol{v}}$ time-independent) Eq. (1.17) yields

$$\bar{\mathbf{p}} = -2\,\eta_\mathrm{iso}\, \overline{\nabla \boldsymbol{v}} + \bar{\mathbf{p}}_\mathrm{T}\,. \tag{4.12}$$

In order to find $\bar{\mathbf{p}}_\mathrm{T}$ needed for (4.12) $\mathbf{p}_\mathrm{T}(t)$ has to be calculated first from Eq. (1.18).

As in (4.2), $\mathbf{p}$ occurring in Eq. (1.18) is replaced by its "isotropic value". Next, the scalar functions $G_m(t)$ are introduced by

$$\begin{aligned} \mathcal{P}^{(m)} : \mathbf{p}_\mathrm{T}(t) &= \mathbf{p}_\mathrm{T}^{(m)}(t) \\ &= -G_m(t)\, A_{\eta\mathrm{T}}\, 2\,\eta_\mathrm{iso}\, \mathcal{P}^{(m)} : \overline{\nabla \boldsymbol{v}}. \end{aligned} \tag{4.13}$$

Then Eq. (1.18) is equivalent to

$$\frac{\partial G_m}{\partial \tau} + [1 + i\,m\,\varphi\ (1 + \beta \sin \Omega\,\tau)]\, G_m = 1, \tag{4.14}$$

where the dimensionless variables

$$\tau = t\,\omega_\mathrm{T}\,, \quad \Omega = \omega/\omega_\mathrm{T} > 0, \quad \varphi = \omega_\mathrm{H}/\omega_\mathrm{T} \gtrless 0 \tag{4.15}$$

are used.

With the ansatz (4.13) the time-averaged viscosity coefficients $\bar{\eta}^{(m)}$ are related to G_m and the field-free viscosity η by

$$\bar{\eta}^{(m)} = \eta[1 + A_{\eta\mathrm{T}}(\bar{G}_m - 1)]\,, \tag{4.16}$$

and one has (for $m = 0, 1, 2$)

$$\bar{\varepsilon}_m \quad = A_{\eta\mathrm{T}}(\mathrm{Re}\, \bar{G}_m - 1)\,, \tag{4.17}$$

$$\bar{\varepsilon}_m^\mathrm{trans} = A_{\eta\mathrm{T}}\, \mathrm{Im}\, \bar{G}_m\,. \tag{4.18}$$

Hence in order to determine these viscosity coefficients, Eq. (4.14) has to be solved for G_m and then time-average has to be performed.

The solution of Eq. (4.14) can be written as

$$G_m(\tau) = \exp\{-\tau - i\,m\,\varphi(\tau - \beta\,\Omega^{-1}\ \cos \Omega\,\tau)\} \cdot \int_0^\tau \mathrm{d}\tau'\, \exp\{\tau' + i\,m\,\varphi\ \ (\tau' - \beta\,\Omega^{-1} \cos \Omega\,\tau')\}\,. \tag{4.19}$$

The initial condition chosen is $G_m(0)=0$. Since it is the long time-behaviour $(\tau\rightarrow\infty)$ which determines the value of the time-average $\bar{G}_m$ the coefficient $\bar{\eta}^{(m)}$ is not affected by the choice of the initial condition for $G_m(\tau)$. Now, use of the formula

$$\exp\{i x\cos y\} = I_0(x) + 2\sum_{l=1}^{\infty} i^l I_l(x)\cos l y\,,\quad I_l(x) = (-1)^l I_l(-x) \tag{4.20}$$

where I_l are Bessel functions of the first kind, and integration over $d\tau'$ leads, for $\tau\rightarrow\infty$, to

$$G_m(\tau) = \left[I_0(m\,a) + 2\sum_{l'=1}^{\infty} i^{l'} I_{l'}(m\,a)\cos l'\,\Omega\,\tau\right]\Bigg\{(1+i\,m\,\varphi)^{-1} I_0(m\,a) + 2\sum_{l=1}^{\infty}(-i)^l I_l(m\,a)$$
$$\cdot\left[(1+i\,m\,\varphi)^2 + l^2\,\Omega^2\right]^{-1}\left[(1+i\,m\,\varphi)\cos l\,\Omega\,\tau + l\,\Omega\sin l\,\Omega\,\tau\right]\Bigg\}\,. \tag{4.21}$$

Here a is an abbreviation for

$$a = \beta\,\varphi/\Omega = \beta\,\omega_{\mathrm{H}}/\omega = \gamma\,H_1/\omega\,, \tag{4.22}$$

where $\omega_{\mathrm{H}} = \gamma\,H$ is the precession frequency of the angular momentum about the applied constant field.

With $\overline{\cos l'\,\Omega\,\tau\cos l\,\Omega\,\tau} = \frac{1}{2}\,\delta_{ll'}$ and $\overline{\cos l'\,\Omega\,\tau\sin l\,\Omega\,\tau} = 0$ the time average

$$\bar{G}_m = (1+i\,m\,\varphi)^{-1} I_0^2(m\,a) + 2\sum_{l=1}^{\infty}(1+i\,m\,\varphi)\,\left[(1+i\,m\,\varphi)^2 + l^2\,\Omega^2\right]^{-1} I_l^2(m\,a) \tag{4.23}$$

is obtained. Hence the magnetic field-induced change of the viscosity coefficients is given by $(m=0,1,2)$

$$\bar{\varepsilon}_m(H,\beta) = A_{\eta\mathrm{T}}\Big\{(1+m^2\varphi^2)^{-1} I_0^2(m\,a) - 1$$
$$+ 2\sum_{l=1}^{\infty}(1+m^2\varphi^2+l^2\,\Omega^2)\,\left[(1-m^2\varphi^2+l^2\,\Omega^2)^2 + 4\,m^2\varphi^2\right]^{-1} I_l^2(m\,a)\Big\} \tag{4.24}$$

and

$$\bar{\varepsilon}_m^{\,\mathrm{trans}}(H,\beta) = -A_{\eta\mathrm{T}}\,m\,\varphi\,\Bigg\{(1+m^2\varphi^2)^{-1} I_0^2(m\,a)$$
$$+ 2\sum_{l=1}^{\infty}(1+m^2\varphi^2-l^2\,\Omega^2)\,\left[(1-m^2\varphi^2+l^2\,\Omega^2)^2 + 4\,m^2\varphi^2\right]^{-1} I_l^2(m\,a)\Bigg\}\,. \tag{4.25}$$

The Eqs. (4.24), (4.25) describing the influence of two collinear static and alternating magnetic fields on the viscosity are the main result of this paper. Some special cases of these formulae are discussed in the next sections. Of course, for $\beta=0$ (no alternating field present) Eqs. (4.24), (4.25) reduce to Eqs. (4.8), (4.9) since $I_0(0)=1$ and $I_l(0)=0$ for $l=1,2,\ldots$.

§ 5. Discussion of Some Special Cases

a) No Constant Field Present

Firstly, it is noticed that in the presence of an alternating field alone ($\mathrm{H}=0$, $\varphi=0$; $\beta\,H = H_1 \neq 0$) $\bar{\varepsilon}_m^{\,\mathrm{trans}}$ vanishes and Eq. (4.24) reduces to

$$\bar{\varepsilon}_m = A_{\eta\mathrm{T}}\left\{I_0^2(m\,\gamma\,H_1/\omega) + 2\sum_{l=1}^{\infty}(1+l^2\,\Omega^2)^{-1} I_l^2(m\,\gamma\,H_1/\omega)\right\}. \tag{5.1}$$

The dimensionless frequency $\Omega = \omega/\omega_{\mathrm{T}}$ is, for constant temperature, inversely proportional to the prespure. It seems instructive to discuss the high and low pressure limits of Eq. (5.1), i. e. the cases $\Omega^2 \ll 1$ and $\Omega^2 \gg 1$. Since

$$I_0^2(x) + 2\sum_{l=1}^{\infty} I_l^2(x) = 1\,,\quad \sum_{l=1}^{\infty} l^2 I_l^2(x) = \tfrac{1}{4}\,x^2\,, \tag{5.2}$$

Eq. (5.1) can be approximated by

$$\bar{\varepsilon}_m \approx A_{\eta\mathrm{T}}\,2\,\Omega^2\sum_{l=1}^{\infty} l^2 I_l^2(m\,\gamma\,H_1/\omega) = \tfrac{1}{2}\,A_{\eta\mathrm{T}}\,\Omega^2\left(\frac{m\,\gamma\,H_1}{\omega}\right)^2 = \tfrac{1}{2}\,A_{\eta\mathrm{T}}\left(\frac{m\,\gamma\,H_1}{\omega_{\mathrm{T}}}\right)^2 \tag{5.3}$$

for $\Omega \ll 1$. Hence the viscosity is not influenced by an alternating magnetic field if the collision frequency ω_{T} is much larger than $m\,\gamma\,H_1$.

In the low pressure regime ($\Omega^2 \gg 1$) where ω is much larger than ω_{T}, Eq. (5.1) can be approximated by

$$\bar{\varepsilon}_m \approx A_{\eta\mathrm{T}} \left\{ I_0{}^2(m\,\gamma\,H_1/\omega) - 1 + \frac{2}{\Omega^2} \sum_{l=1}^{\infty} l^{-2}\, I_l{}^2(m\,\gamma\,H_1/\omega) \right\}. \tag{5.4}$$

If the term proportional to Ω^{-2} can also be neglected, $-\bar{\varepsilon}_m$ as a function of $m\,|\gamma|\,H_1/\omega$ reaches its first maximum at

$$m\,|\gamma|\,H_1/\omega = j_{01} \tag{5.5}$$

where $j_{01} \approx 2.4$ is the first zero of the Bessel function I_0. Hence such a measurement of $\bar{\varepsilon}_m$ would allow a determination of the gyromagnetic ratio $|\gamma|$.

b) $\bar{\varepsilon}_m$ and $\bar{\varepsilon}^m_{\mathrm{trans}}$ as Functions of φ

Next, $\bar{\varepsilon}_m$ and $\bar{\varepsilon}_m^{\mathrm{trans}}$ are considered as functions of H or φ for a fixed frequency and magnitude of the alternating field. Again, it is interesting to discuss the high and low pressure limits of these equations. For $\Omega^2 \ll 1$ (high pressure) Eqs. (4.24), (4.25) can be approximated by [for a see (4.22)]

$$\bar{\varepsilon}_m \approx A_{\eta\mathrm{T}} \left\{ -\frac{m^2\,\varphi^2}{1+m^2\,\varphi^2} + \frac{1}{2}\left(\frac{m\,\gamma\,H_1}{\omega_{\mathrm{T}}}\right)^2 \frac{1+m^4\,\varphi^4}{(1+m^2\,\varphi^2)^4} \right\}, \tag{5.6}$$

$$\bar{\varepsilon}_m^{\mathrm{trans}} \approx -A_{\eta\mathrm{T}}\, m\,\varphi \left\{ (1+m^2\,\varphi^2)^{-1} + \frac{1}{2}\left(\frac{m\,\gamma\,H_1}{\omega_{\mathrm{T}}}\right)^2 \frac{1+4\,m^2\,\varphi^2+m^4\,\varphi^4}{(1+m^2\,\varphi^2)^4} \right\}. \tag{5.7}$$

Hence if the frequency $m\,|\gamma|\,H_1$ is very small compared with the relaxation frequency, ω_{T}, the alternating field does not affect the viscosity. More interesting is the low pressure limit $\Omega^2 \gg 1$. In this case one has

$$\bar{\varepsilon}_m \approx A_{\eta\mathrm{T}} \left\{ (1+m^2\,\varphi^2)^{-1}\, I_0{}^2(m\,a) - 1 + \sum_{l=1}^{\infty} h_{lm}\, I_l{}^2(m\,a) \right\}, \tag{5.8}$$

with

$$h_{lm} = 2\,(l^2\,\Omega^2 + m^2\,\varphi^2)\,[(l^2\,\Omega^2 - m^2\,\varphi^2)^2 + 4\,m^2\,\varphi^2]^{-1}, \tag{5.9}$$

and

$$\bar{\varepsilon}_m^{\mathrm{trans}} \approx -A_{\eta\mathrm{T}}\, m\,\varphi \left\{ (1+m^2\,\varphi^2)^{-1}\, I_0{}^2(m\,a) + \sum_{l=1}^{\infty} k_{lm}\, I_l{}^2(m\,a) \right\}, \tag{5.10}$$

with

$$k_{lm} = 2\,(m^2\,\varphi^2 - l^2\,\Omega^2)\;[(l^2\,\Omega^2 - m^2\,\varphi^2)^2 + 4\,m^2\,\varphi^2]^{-1}. \tag{5.11}$$

For $m\,|\varphi| \approx l\,\Omega$ the function h_{lm} can be approximated by a Lorentzian and k_{lm} by a "dispersion" curve, i. e. one has

$$h_{lm} \approx [(l\,\Omega - m\,|\varphi|)^2 + 1]^{-1}, \quad k_{lm} \approx \frac{1}{l\,\Omega}\,(m\,|\varphi| - l\,\Omega)\;[(l\,\Omega - m\,|\varphi|)^2 + 1]^{-1}. \tag{5.12}$$

Thus, according to (5.8), (5.9), (5.12) the quantity $-A_{\eta\mathrm{T}}^{-1}\,\bar{\varepsilon}_m$ plotted as function of $m\,|\varphi|$ with constant a reaches its saturation value 1 for $m\,|\varphi| \gg 1$ except for $m\,|\varphi| = l\,\Omega$, $l = 1, 2, 3, \ldots$ where the curve shows "dips" of width 1 and of depth $I_l{}^2(m\,\gamma\,H_1/\omega)$. Since one has at the position of the first dip $(m\,\omega_{\mathrm{H}})_{\mathrm{dip}} = m\,\gamma\,H_{\mathrm{dip}} = \omega$, the magnitude of the gyromagnetic ratio γ can be determined according to

$$|\gamma| = \omega/(m\,H_{\mathrm{dip}}). \tag{5.13}$$

Notice that $\bar{\varepsilon}_m$ and $\bar{\varepsilon}_m^{\mathrm{trans}}$ as given by (4.21), (4.22) or the approximate equations (5.8), (5.10) display a pressure dependence via $\Omega = \omega/\omega_{\mathrm{T}}$ when plotted as functions of φ or of the ratio H/p_0. Such a pressure dependence also shows up when the influence of a constant magnetic field on the viscosity and the heat conductivity of O_2 is investigated at low pressures[27]. This behaviour may qualitatively be understood when it is assumed that the coupling between the electronic spin and the rotational angular momentum is similar to an "internal" oscillating magnetic field.

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[27] H. Hulsman, A. L. J. Burgmans, E. J. van Waasdijk, and H. F. P. Knaap, Physica **50**, 558 [1970].

§ 6. Frequency-dependence of the Viscosity Coefficients and its Relevance for the Dynamic Thermomagnetic Torque

The relative change of the viscosity coefficients as given by Eqs. (4.24), (4.25) depends on the frequency ω of the alternating field via $\alpha = \gamma H_1/\omega = \beta\, \omega_{\mathrm{H}}/\omega$ (which is the argument of the Bessel functions) and via $\Omega = \omega/\omega_{\mathrm{T}}$ occurring in the factors in front of the I_l^2 (with $l \geq 1$). Two dimensionless frequency variables Ω or $x = \omega/m\, \omega_{\mathrm{H}} = \Omega/m\, \varphi$ can be used to discuss the frequency dependence described by Eqs. (4.24), (4.25). For $m\varphi > 1$ when the angular momentum of a molecule makes more than one precession between two collisions, x will turn out to be the more convenient variable, but for $m\varphi < 1$ it is more convenient to use Ω.

Now, Eq. (4.24) is equivalent to

$$\frac{\bar{\varepsilon}_m(H,\beta)}{\bar{\varepsilon}_m(H,0)} - 1 = \frac{1}{m^2\varphi^2}\,[1 - I_0^2(m\,\beta\,\varphi/\Omega)] - \frac{1+m^2\varphi^2}{m^2\varphi^2}\; 2\sum_{l=1}^{\infty} \frac{1+m^2\varphi^2+l^2\Omega^2}{[1-m^2\varphi^2+l^2\Omega^2]^2+4\,m^2\varphi^2}\, I_l^2(m\,\beta\,\varphi/\Omega) \tag{6.1}$$

$$= (m\,\varphi)^{-2}[1 - I_0^2(\beta/x)] - \frac{1+m^2\varphi^2}{m^4\varphi^4}\; 2\sum_{l=1}^{\infty} \frac{1+(m\,\varphi)^{-2}+l^2x^2}{[l^2x^2-1+(m\,\varphi)^{-2}]^2+4(m\,\varphi)^{-2}}\, I_l^2(\beta/x), \tag{6.2}$$

with

$$x = \omega/m\,\omega_{\mathrm{H}}\,, \qquad m = 1, 2\,. \tag{6.3}$$

Likewise (4.25) is equivalent to

$$\frac{\bar{\varepsilon}_m^{\mathrm{trans}}(H,\beta)}{\bar{\varepsilon}_m^{\mathrm{trans}}(H,0)} - 1 = I_0^2(\beta\, m\,\varphi/\Omega) - 1 + 2(1+m^2\varphi^2) \sum_{l=1}^{\infty} \frac{1+m^2\varphi^2-l^2\Omega^2}{[1-m^2\varphi^2+l^2\Omega^2]^2+4\,m^2\varphi^2}\, I_l^2(m\,\beta\,\varphi/\Omega) \tag{6.4}$$

$$= I_0^2(\beta/x) - 1 + 2[1+(m\,\varphi)^{-2}] \sum_{l=1}^{\infty} \frac{1+(m\,\varphi)^{-2}-l^2x^2}{[l^2x^2-1+(m\,\varphi)^{-2}]^2+4(m\,\varphi)^{-2}}\, I_l^2(\beta/x)\,. \tag{6.5}$$

Eqs. (6.4) and (6.5) are of particular importance for the "dynamic" thermomagnetic torque effect[3]. Within certain approximations discussed in Ref. [4] the difference ΔN between the torque measured with and witlout oscillating magnetic field divided by the torque N for the constant magnetic field alone is given by

$$\Delta N/N = \bar{\varepsilon}_2^{\mathrm{trans}}(H,\beta)/\bar{\varepsilon}_2^{\mathrm{trans}}(H,0) - 1\,. \tag{6.6}$$

Curves for $\Delta N/N$ evaluated according to (6.5), (6.6) have been shown in Ref. [4] for the case $(m\,\varphi)^2 > 1$.

It seems useful to consider Eqs. (6.4), (6.5) for the limiting cases $(m\,\varphi)^2 \ll 1$ and $(m\,\varphi)^2 \gg 1$ in more detail. For $(m\,\varphi)^2 \ll 1$ in Eq. (6.4) can be approximated by

$$\frac{\bar{\varepsilon}_m^{\mathrm{trans}}(H,\beta)}{\bar{\varepsilon}_m^{\mathrm{trans}}(H,0)} - 1 = I_0^2(\beta\, m\,\varphi/\Omega) - 1 + 2\sum_{l=1}^{\infty} \frac{1-l^2\Omega^2}{(1+l^2\Omega^2)^2}\, I_l^2(\beta\, m\,\varphi/\Omega)\,. \tag{6.7}$$

If Ω is not too small, i. e. if

$$\beta\, m\,\varphi/\Omega \lesssim 0.3 \tag{6.8}$$

it is sufficient to retain the Bessel function I_1 in the sum occurring in (6.7) and to truncate the power series expansion of I_0^2 and I_1^2 after the 2nd and 1st term, respectively. Then Eq. (6.7) reduces to

$$\frac{\bar{\varepsilon}_m^{\mathrm{trans}}(H,\beta)}{\bar{\varepsilon}_m^{\mathrm{trans}}(H,0)} - 1 \approx -\frac{3}{2}\,\beta^2 m^2 \varphi^2 F(\Omega) \tag{6.9}$$

with

$$F(\Omega) = \tfrac{1}{3}\,(1+\Omega^2)^{-1} + \tfrac{2}{3}\,(1+\Omega^2)^{-2}\,. \tag{6.10}$$

In Ref. [3] the difference ΔN of the torque with and without alternating magnetic field divided by ΔN for $\omega \rightarrow 0$ has been measured and plotted for the case $(m\,\varphi)^2 \ll 1$. According to (6.6) this normalized quantity $(\Delta N)_{\mathrm{norm}}$ is equal to $F(\Omega)$. Due to (6.10) $F(\Omega)$ approaches 1 and 0, for $\Omega \rightarrow 0$ and

$\Omega \to \infty$, respectively, and one has $F = \frac{1}{2}$ for

$$\Omega_{1/2}^2 = (5-\sqrt{13})/(\sqrt{13}-1) , \text{ or } \quad \Omega_{1/2} \approx 0.73 . \tag{6.11}$$

Hence the frequency $\omega_{1/2}$ for which $(\Delta N)_{\text{norm}}$ as plotted in Ref. [3] assumes the value 1/2 is related to the relaxation frequency of the tensor polarization ω_{T} by

$$\omega_{1/2} \approx 0.73 \; \omega_{\text{T}} . \tag{6.12}$$

Since ω_{T} is proportional to the pressure (for constant temperature) $\omega_{1/2}$ should increase proportional to the pressure. This was indeed observed experimentally [3].

Finally the case $(m\,\varphi)^2 \gg 1$ is considered. In this case Eq. (6.5) can be approximated by

$$\frac{\bar{\varepsilon}_m^{\text{trans}}(H,\beta)}{\bar{\varepsilon}_m^{\text{trans}}(H,0)} - 1 \approx I_0^2(\beta/x) - 1 + 2 \sum_{l=1}^{\infty} \frac{1-l^2 x^2}{[1-l^2 x^2]^2 + 4(m\,\varphi)^{-2}} \, I_l^2(\beta/x) . \tag{6.13}$$

The factor in front of I_l^2 describes a dispersion like curve which approaches the values 1 and 0 for $x \to 0$ and $x \to \infty$ respectively, it vanishes for the "dispersion frequency" $x = l^{-1}$, and it assumes its maximum and minimum values $\pm \frac{1}{2}(m\,\varphi)^{-1}$ for $x \approx l^{-1}\,(1 \mp (m\,\varphi)^{-1})$. Thus Eq. (6.13) describes a set of dispersion-like curves which sit on a "background" determined by $I_0^2(\beta/x) - 1$. Again the experimental results reported in Ref. [3] for $\Delta N/N$ measured as function of the frequency for experimental situations where $(m\,\varphi)^2 \gtrsim 5$, are qualitatively of this type. Since one has for the first dispersion frequency $x^2 = 1$ or

$$\omega = m\,|\,\omega_{\text{H}}\,| = m\,|\,\gamma\,|\,H \tag{6.14}$$

it is possible to determine the value of the gyromagnetic ratio $|\gamma|$ by such a measurement. From the thermomagnetic torque effect (where $m = 2$ applies) values of $|\gamma|$ have been obtained by SMITH and SCOTT [3] which are in good agreement with values of $|\gamma|$ determined by other methods.

Appendix

It is the purpose of this Appendix to discuss the connection of the 4th-rank tensor $\mathcal{H}$ and of the projection tensors $\mathcal{P}^{(m)}$ [cf. (1.14) and (3.2), (3.3)] with the active rotation of a 2nd rank tensor about a given direction and with the transformation of the components of a 2nd-rank tensor under a rotation of the coordinate system. Furthermore, the physical meaning of the viscosity coefficients as introduced in § 3 is studied, i. e. it is indicated how the various coefficients can be measured. Firstly, however, the active rotation of a vector and the transformation of its components under a rotation of the coordinate system are discussed.

1. Rotation of a Vector, Projection Tensors of Rank 2

An active infinitesimal rotation of a vector $\boldsymbol{a}$ by a small angle ϑ about a direction specified by the unit vector $\boldsymbol{h}$ leads to the vector

$$\boldsymbol{a}' = \boldsymbol{a} + \vartheta\,\mathbf{H} \cdot \boldsymbol{a} \tag{A.1}$$

where the 2nd rank tensor $\mathbf{H}$ is given by

$$H_{\mu\nu} = \varepsilon_{\mu\lambda\nu}\,h_\lambda . \tag{A.2}$$

A rotation by a finite angle ϑ yields

$$\boldsymbol{a}' = (e^{\vartheta \mathbf{H}}) \cdot \mathbf{a} = \mathbf{R}(\vartheta) \cdot \boldsymbol{a} . \tag{A.3}$$

Since $\mathbf{H}^2 = \mathbf{H} \cdot \mathbf{H} = \boldsymbol{h}\,\boldsymbol{h} - \mathbf{1}$ the tensor $\mathbf{H}$ obeys the equation

$$\mathbf{H}^3 + \mathbf{H} = 0 . \tag{A.4}$$

Eq. (A.4) can be considered as a Hamilton-Cayley equation for the "tensor-operator" $\mathbf{H}$ with the eigenvalues $i\,m$, $m = 0, \pm 1$. Thus it is possible to define "projection tensors" $\mathbf{P}^{(m)}$ $(m = 0, \pm 1)$ by

$$\mathbf{P}^{(m)} = \prod_{m' \neq m} \frac{\mathbf{H} - i\,m'}{i\,m - i\,m'} \, ; \quad m, m' = 0, \pm 1 . \tag{A.5}$$

Explicitly, one has

$$\mathbf{P}^{(0)} = \boldsymbol{h}\,\boldsymbol{h} ; \qquad \mathbf{P}^{(\pm 1)} = \tfrac{1}{2}\,(\mathbf{1} - \boldsymbol{h}\,\boldsymbol{h} \mp i\,\mathbf{H}) . \tag{A.6}$$

These projectors have the properties

$$\mathbf{P}^{(m)} \cdot \mathbf{P}^{(m')} = \mathbf{P}^{(m)}\,\delta_{mm'} ; \qquad \mathbf{P}^{(m)*} = \mathbf{P}^{(-m)} , \tag{A.7}$$

$$\sum_{m=-1}^{+1} \mathbf{P}^{(m)} = \mathbf{1} , \quad P_{\mu\mu}^{(m)} = 1 , \tag{A.8}$$

$$\mathbf{P}^{(m)} \cdot \mathbf{H} = \mathbf{H} \cdot \mathbf{P}^{(m)} = i\,m\,\mathbf{P}^{(m)} . \tag{A.9}$$

Due to (A.8), (A.9) the tensor $\mathbf{H}$ can be resolved in terms of its eigentensors $\mathbf{P}^{(m)}$ according to

$$\mathbf{H} = \sum_{m=-1}^{+1} i\, m\, \mathbf{P}^{(m)} . \tag{A.10}$$

The tensor $R(\vartheta)$ describing the active rotation of a vector by the angle ϑ can be written as

$$\mathbf{R}(\vartheta) = e^{\vartheta \mathbf{H}} = \sum_{m=-1}^{+1} \mathbf{P}^{(m)} \cdot e^{\vartheta \mathbf{H}} = \sum_{m=-1}^{+1} e^{i m \vartheta}\, \mathbf{P}^{(m)} \tag{A.11}$$

or

$$\mathbf{R}(\vartheta) = \mathbf{P}^{(0)} + \cos\vartheta\, (\mathbf{P}^{(1)} + \mathbf{P}^{(-1)}) + \sin\vartheta\, i\, (\mathbf{P}^{(1)} - \mathbf{P}^{(-1)}) . \tag{A.12}$$

Next, a few remarks on the transformation of the Cartesian components of a vector under the rotation of the coordinate system are in order. Let $\mathbf{U}(\vartheta)$ be the orthogonal transformation matrix which transforms the Cartesian components of a vector in the original system to the components in a coordinate system rotated by the angle ϑ about a direction specified by $\boldsymbol{h}$. Now, if a vector $\boldsymbol{a}$ is first rotated by an angle and if then the coordinate system is rotated by the same angle the Cartesian components of the rotated vector in the rotated system are the same as that of $\boldsymbol{a}$ in the original coordinate system or

$$a_\mu = U_{\mu\nu}(\vartheta)\, a_\nu' = U_{\mu\nu}(\vartheta)\, R_{\nu\lambda}(\vartheta)\, a_\lambda . \tag{A.13}$$

Thus one has

$$U_{\mu\nu}(\vartheta)\, R_{\nu\lambda}(\vartheta) = \delta_{\mu\nu}$$

or

$$U_{\mu\nu}(\vartheta) = R_{\mu\nu}(-\vartheta) = R_{\nu\mu}(\vartheta) . \tag{A.14}$$

The transformation matrix $U_{\mu\nu}(\vartheta)$ can also be written in terms of the tensors $P_{\mu\nu}^{(m)}$. This point is of particular interest if one wants to define the coefficients characterizing the 2nd-rank tensor which links two vectors in the presence of a magnetic field parallel to $\boldsymbol{h}$ in such a way that they remain invariant under a rotation of the coordinate system about $\boldsymbol{h}$. For the heat conductivity tensor, e. g. it is well-known that the coefficients $\lambda_{\|}$, $\lambda_{\perp}$ and λ_{trans} defined by

$$\begin{aligned} \boldsymbol{\lambda} &= \lambda_{\|}\, \boldsymbol{h}\boldsymbol{h} + \lambda_{\perp} (\mathbf{1} - \boldsymbol{h}\boldsymbol{h}) + \lambda_{\text{trans}}\, \mathbf{H} \\ &= \lambda_{\|}\, \mathbf{P}^{(0)} + \lambda_{\perp} (\mathbf{P}^{(1)} + \mathbf{P}^{(-1)}) \\ &\quad + \lambda_{\text{trans}}\, i (\mathbf{P}^{(1)} - \mathbf{P}^{(-1)}) \end{aligned} \tag{A.15}$$

are of this type. Notice that this is a decomposition of a 2nd-rank tensor similar to (A.12). In analogy to (A.11) one may introduce complex heat conductivity coefficients $\lambda^{(m)}$

$$\boldsymbol{\lambda} = \sum_{m=-1}^{+1} \lambda^{(m)}\, \mathbf{P}^{(m)} \tag{A.16}$$

which are related to $\lambda_{\|}$, $\lambda_{\perp}$, λ_{trans} by

$$\lambda^{(0)} = \lambda_{\|} , \quad \lambda^{(\pm 1)} = \lambda_{\perp} \pm i\, \lambda_{\text{trans}} . \tag{A.17}$$

(The subscripts "$\|$", "$\perp$", "trans" refer to parallel, perpendicular, and transverse with respect to the direction specified by $\boldsymbol{h}$.)

A decomposition of the viscosity tensor corresponding to (A.15) or (A.16) and the definition of viscosity coefficients which are invariant under a rotation of the coordinate system about $\boldsymbol{h}$ can be obtained by studying the rotation of a 2nd rank tensor.

2. *Rotation of a 2nd Rank Tensor, Projection Tensors of Rank 4*

Let $\mathbf{A} = \boldsymbol{a}\,\boldsymbol{a}$ be the 2nd rank tensor constructed from the vector $\boldsymbol{a}$. An infinitesimal rotation of $\boldsymbol{a}$ as described by (A.1) leads to

$$\begin{aligned} A_{\mu\nu}' &= A_{\mu\nu} + \vartheta\, (H_{\mu\mu'}\, \delta_{\nu\nu'} + \delta_{\mu\mu'}\, H_{\nu\nu'})\, A_{\mu'\nu'} \\ &= A_{\mu\nu} + \vartheta\, \mathcal{H}_{\mu\nu,\, \mu'\nu'}\, A_{\mu'\nu'} . \end{aligned} \tag{A.18}$$

Hence the 4th rank tensor $\mathcal{H}$ which already occurred in Eqs. (1.9), (1.14), (1.15) is the generator of the infinitesimal rotation of a 2nd rank tensor. Now, using (A.8) and (A.10) one may write $\mathcal{H} : \mathbf{A}$ in the form

$$\mathcal{H}_{\mu\nu,\, \mu'\nu'}\, A_{\mu'\nu'} = \sum_{m_1=-1}^{+1} \sum_{m_2=-1}^{+1} i (m_1 + m_2)\, \mathcal{P}_{\mu\nu,\mu'\nu'}^{(m_1 m_2)}\, A_{\mu'\nu'} \tag{A.19}$$

with

$$\mathcal{P}_{\mu\nu,\mu'\nu'}^{(m_1 m_2)} = P_{\mu\mu'}^{(m_1)}\, P_{\nu\nu'}^{(m_2)} \tag{A.20}$$

or

$$\mathcal{P}^{(m_1 m_2)} = P^{(m_1)} \times P^{(m_2)} \text{ (Kronecker product)}.$$

From definition (A.20) and the properties (A.7) follows

$$\mathcal{P}^{(m_1 m_2)} : \mathcal{P}^{(m_1' m_2')} = \mathcal{P}^{(m_1 m_2)}\, \delta_{m_1 m_1'}\, \delta_{m_2 m_2'} . \tag{A.20a}$$

Replacing the $\mathbf{A}$ in Eq. (A.19) by one of the $\mathcal{P}$'s gives

$$\mathcal{H} : \mathcal{P}^{(m_1 m_2)} = i (m_1 + m_2)\, \mathcal{P}^{(m_1 m_2)} . \tag{A.21}$$

Hence $\mathcal{H}$ has the 5 eigenvalues $i\, m$ with $m = 0, \pm 1, \pm 2$ since $m = m_1 + m_2$ and $m_1, m_2 = 0, \pm 1$. Therefore $\mathcal{H}$ obeys the Hamilton-Cayley equation

$$\prod_{m=-2}^{+2} (\mathcal{H} - i\, m) = 0 . \tag{A.22}$$

The 5 associated, linearly independent eigentensors are the projection tensors

$$\mathcal{P}_{\mu\nu,\mu'\nu'}^{(m)} = \sum_{m_1} \sum_{m_2} \delta_{m,\, m_1 + m_2}\, P_{\mu\mu'}^{(m_1)}\, P_{\nu\nu'}^{(m_2)} . \tag{A.23}$$

Summed they give the unity tensor,

$$\sum_{m} P^{(m)}_{\mu\nu,\mu'\nu'} = \sum_{m_1} \sum_{m_2} P^{(m_1)}_{\mu\mu'} \mathcal{P}^{(m_2)}_{\nu\nu'} = \delta_{\mu\mu'}\, \delta_{\nu\nu'}$$

after (A.8).

Eq. (A.19) can be rewritten as

$$\mathcal{H} = \sum_{m=-2}^{+2} i\, m\, \mathcal{P}^{(m)}\,. \tag{A.24}$$

The projection tensors $\mathcal{P}^{(m)}$ have been used in § 3.

Analogous to (A.5) the $\mathcal{P}^{(m)}$ can also be expressed in terms of the 4th rank tensor $\mathcal{H}$ by

$$\mathcal{P}^{(m)} = \prod_{m' \neq m} \frac{\mathcal{H} - i\, m'}{i\, m - i\, m'}\,. \tag{A.25}$$

As irreducibility is conserved in rotation, the same is true for application of $\mathcal{H}$ and of the $\mathcal{P}^{(m)}$, or in a formula

$$\mathcal{H} : \overline{\mathbf{A}} = \overline{\mathcal{H} : \mathbf{A}}; \quad \mathcal{P}^{(m)} : \overline{\mathbf{A}} = \overline{\mathcal{P}^{(m)} : \mathbf{A}}\,. \tag{A.26}$$

Here $\overline{\mathbf{A}}$ denotes the irreducible part of the 2nd rank tensor **A**.

Now, we are ready to turn to a discussion of the rotation of the 2nd rank tensor **A** by a finite angle ϑ. In correspondence with (A.3) one has

$$\mathbf{A}' = (e^{\vartheta \mathcal{H}}) : \mathbf{A} = \mathcal{R}(\vartheta) : \mathbf{A}\,. \tag{A.27}$$

Similar to (A.11) or (A.12) the 4th rank rotation tensor $\mathcal{R}(\vartheta)$ can be resolved as

$$\mathcal{R}(\vartheta) = \sum_{m=-2}^{+2} \mathcal{P}^{(m)} : (e^{\vartheta \mathcal{H}}) = \sum_{m=-2}^{+2} e^{i m \vartheta}\, \mathcal{P}^{(m)}\,, \tag{A.28}$$

or

$$\mathcal{R}(\vartheta) = \mathcal{P}^{(0)} + \sum_{m=1}^{2} [\cos(m\,\vartheta)\, (\mathcal{P}^{(m)} + \mathcal{P}^{(-m)}) + \sin(m\,\vartheta)\; i (\mathcal{P}^{(m)} - \mathcal{P}^{(-m)})]\,. \tag{A.29}$$

The definition of the viscosity coefficients introduced in (3.1) and (3.7) is based on a decomposition of the viscosity tensor similar to the decomposition of $\mathcal{R}(\vartheta)$ given by (A.28) and (A.29). The coefficients $\eta^{(m)}$ and ε_m, $\varepsilon_m^{\text{trans}}$ are invariant under a rotation of the coordinate system about a direction parallel to $\boldsymbol{h}$ because $\mathcal{R}(\vartheta)$ (or $\mathcal{H}$) commutes with the η-tensor. The fact that there are 5 linearly independent viscosity coefficients in the presence of a magnetic field is due to the fact that there are 5 projection tensors $\mathcal{P}^{(m)}$ corresponding to the 5 eigenvalues of the 4th rank tensor $\mathcal{H}$.

3. *The Friction Pressure Tensor*

The irreducible second rank pressure tensor is

$$p_{\mu\nu} = -2 \sum_{-2}^{2} \eta^{(m)}\, P^{(m)}_{\mu\nu,\mu'\nu'}\, \overline{\nabla v}_{\mu'\nu'}\,, \tag{A.30}$$

with the abbreviation

$$\overline{\nabla v}_{\mu\nu} = \frac{1}{2}\left(\frac{\partial v_\nu}{\partial x_\mu} + \frac{\partial v_\mu}{\partial x_\nu}\right) - \frac{1}{3}\, \delta_{\mu\nu}\, \frac{\partial v_\lambda}{\partial x_\lambda}\,.$$

Use of (A.23) and carrying out the m-summation gives instead of (A.30)

$$p_{\mu\nu} = -2 \sum_{-1}^{1} \sum_{-1}^{1} \eta^{(m_1+m_2)}\, P^{(m_1)}_{\mu\mu'}\, P^{(m_2)}_{\nu\nu'}\, \overline{\nabla v}_{\mu'\nu'}\,. \tag{A.31}$$

In tensor notation (without subscripts) this reads

$$\mathbf{p} = -2 \sum \sum \eta^{(m_1+m_2)}\, \mathbf{P}^{(m_1)} \cdot \overline{\nabla \boldsymbol{v}} \cdot \mathbf{P}^{(-m_2)}. \tag{A.32}$$

Again, introducing m-components of the nabla and velocity vectors

$$\nabla^{(m)}_{\mu} = P^{(m)}_{\mu\mu'}\, \nabla_{\mu'}\,, \quad v^{(m)}_{\mu} = P^{(m)}_{\mu\mu'}\, v_{\mu'}\,, \quad m = 0, \pm 1,$$

and using the properties of the projectors **P**, one can rewrite (A.31) as follows

$$p_{\mu\nu} = -\sum \sum \eta^{(m_1+m_2)} (\nabla^{(m_1)}_{\mu}\, v^{(m_2)}_{\nu} + \nabla^{(m_1)}_{\nu}\, v^{(m_2)}_{\mu}) + \tfrac{2}{3}\, \eta^{(0)}\, \delta_{\mu\nu}\, \nabla_\lambda v_\lambda\,. \tag{A.33}$$

In the Navier-Stokes equation the force-density

$$f_\mu = -\nabla_\nu\, p_{\nu\mu}\,, \tag{A.34}$$

the divergence of the friction tensor, appears. Its m-component

$$f^{(m)}_{\mu} = P^{(m)}_{\mu\mu'}\, f_{\mu'}\,, \quad f_\mu = \sum_{-1}^{1} f^{(m)}_{\mu} \tag{A.35}$$

taken from (A.32), is

$$f^{(m)}_{\mu} = \sum_{m_1=-1}^{1} \eta^{(m+m_1)}\, \triangle^{(m_1)}\, v^{(m)}_{\mu} + \nabla^{(m)}_{\mu} \sum_{m_1=-1}^{1} (\eta^{(m+m_1)} - \tfrac{2}{3}\, \eta^{(0)})\, \nabla^{(-m_1)}_{\nu}\, v^{(m_1)}_{\nu} \tag{A.36}$$

where

$$\triangle^{(m_1)} = \nabla^{(-m_1)}_{\nu}\, \nabla^{(m_1)}_{\nu}$$

is, so to say, the m_1-part of the Laplacian. The viscosity has been treated as spatially constant. In the isotropic case

$$\eta^{(m)} = \eta^{(0)} = \eta \quad \text{for} \quad m = -2, \dots, +2,$$

(A.36) yields the familiar expression

$$f_\mu = \eta\, (\Delta v_\mu + \nabla_\mu \nabla_\nu v_\nu)\,.$$

4. *The Two-dimensional Case*

The "part of the friction pressure tensor perpendicular to the magnetic field" has to be defined by

$$\mathbf{p}_\perp = (1 - \mathbf{P}^{(0)}) \cdot \mathbf{p} \cdot (1 - \mathbf{P}^{(0)})\,. \tag{A.37}$$

As

$$(1-\mathbf{P}^{(0)})\cdot\mathbf{P}^{(m)}=\mathbf{P}^{(m)}(1-\mathbf{P}^{(0)})=(1-\delta_{m0})\,\mathbf{P}^{(m)}$$

one obtains from (A.32)

$$\mathbf{p}_{\perp}=-2\sum{}'\sum{}'\,\eta^{(m_1+m_2)}\,\mathbf{P}^{(m_1)}\cdot\overline{\nabla\boldsymbol{v}}\cdot\mathbf{P}^{(-m_2)}\,,$$

where the summation goes over $m_{1,2}\neq 0$. In extenso, one has with $\mathbf{S}=\overline{\nabla\boldsymbol{v}}$

$$\mathbf{p}_{\perp}=-2\,[\eta^{(-2)}\,\mathbf{P}^{(-1)}\cdot\mathbf{S}\cdot\mathbf{P}^{(1)}+\eta^{(0)}(\boldsymbol{P}^{(-1)}\cdot\mathbf{S}\cdot\boldsymbol{P}^{(-1)} +\mathbf{P}^{(1)}\cdot\mathbf{S}\cdot\mathbf{P}^{(1)})+\eta^{(2)}\,\mathbf{P}^{(1)}\cdot\mathbf{S}\cdot\mathbf{P}^{(-1)}]. \tag{A.38}$$

Now let us denote the Cartesian components in the plane perpendicular to the magnetic field by Latin subscripts k, l, etc., having the values 1 and 2. The relevant components of the projectors are

$$P_{kk'}^{(\pm 1)}=\tfrac{1}{2}\,(\delta_{kk'}\pm i\,\varepsilon_{kk'})$$

with the two-dimensional ε-tensor

$$\varepsilon=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\,;\quad \boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}=-\,\boldsymbol{\delta}\,.$$

By use of the formula

$$\varepsilon_{kk'}\,\varepsilon_{ll'}=\delta_{kl}\,\delta_{k'l'}-\delta_{kl'}\,\delta_{k'l}$$

one obtains after a little algebra from (A.38)

$$\begin{aligned}(p_{\perp})_{kl}=\;&-\mathrm{Re}\,\eta^{(2)}(\nabla_k v_l+\nabla_l v_k-\delta_{kl}\nabla_j v_j)\\&-\mathrm{Im}\,\eta^{(2)}\,\tfrac{1}{2}\,[\varepsilon_{kj}(\nabla_j v_k+\nabla_l v_j)\\&+\varepsilon_{lj}(\nabla_j v_k+\nabla_k v_j)]\\&-\tfrac{1}{3}\,\eta^{(0)}\,\delta_{kl}(\nabla_j v_j-2\,\nabla_3 v_3)\,.\end{aligned}\tag{A.39}$$

The real part of $\eta^{(2)}$ is multiplied by twice the shear tensor irreducible in two dimensions. The $\eta^{(0)}$-term vanishes if the flow is non-divergent and independent on x_3, the coordinate in the magnetic field direction; in that case $\mathbf{p}_{\perp}$ is irreducible in two dimensions.

If the flow pattern does not depend on x_3, the force density, derived from (A.39), is

$$\begin{aligned}f_k&=-\nabla_l(p_{\perp})_{kl}\\&=\mathrm{Re}\,\eta^{(2)}\,\Delta v_k+\mathrm{Im}\,\eta^{(2)}\,\varepsilon_{kl}\,\Delta v_l\\&\quad+\tfrac{1}{3}\,\eta^{(0)}\,\nabla_k\nabla_l v_l\,.\end{aligned}\tag{A.40}$$

Here, $\Delta=\nabla_l\nabla_l$ denotes the two-dimensional Laplacian. The obvious formula (non-existence of a completely antisymmetric 3rd rank tensor in two dimensions)

$$\varepsilon_{jk}\,\nabla_l+\varepsilon_{lj}\,\nabla_k+\varepsilon_{kl}\,\nabla_j\equiv 0$$

has been used in the derivation.

5. *On the Measurement of the Viscosity Coefficients*

To find the physical meaning of the viscosity coefficients the hydrodynamical steady state flow problem has to be solved for a specific experimental set-up. However, for the Senftleben-Beenakker effect the deviation of the viscosity tensor in the presence of a magnetic field $\boldsymbol{H}$ from the isotropic viscosity tensor for $H=0$ is small (i. e. $|\varepsilon_m|\ll 1$; $|\varepsilon_m^{\text{trans}}|\ll 1$). Thus, in first approximation, the velocity field can be assumed to be unaffected by the magnetic field. Then application of a magnetic field gives rise to a change $\delta p=p(\mathrm{H})-p(0)$ of the static pressure p which can be determined from

$$\delta\mathbf{f}\equiv\nabla\delta p=2\,\eta\,\nabla\cdot\left\{\varepsilon^{(0)}\,\mathcal{P}^{(0)}+\sum_{m=1}^{2}[\varepsilon_m(\mathcal{P}^{(m)}+\mathcal{P}^{(-m)}) +\varepsilon_m^{\text{trans}}\,i(\mathcal{P}^{(m)}-\mathcal{P}^{(-m)})]\right\}:\overline{\nabla\boldsymbol{v}} \tag{A.41}$$

where $\boldsymbol{v}$ is taken as the flow velocity in the absence of a magnetic field.

As a special case the flow through a tube with a flat rectangular cross section is considered. It is assumed that the velocity $\boldsymbol{v}(\boldsymbol{x})$ can be approximated by that of the flow between flat plates.

Let $\hat{\boldsymbol{v}}$ and $\boldsymbol{n}$ be unit vectors parallel to $\boldsymbol{v}$ and perpendicular to the flat plates, respectively. The flow velocity is parallel to the plates (i. e. $\boldsymbol{n}\cdot\boldsymbol{v}=0$) and it depends on $\boldsymbol{x}$ only via $y=\boldsymbol{n}\cdot\boldsymbol{x}$. Thus one may write $\boldsymbol{v}(\boldsymbol{x})=v(y)\,\hat{\boldsymbol{v}}$ and $\nabla\,\overline{\nabla\boldsymbol{v}}$ occurring in Eq. (A.41) is given by

$$\nabla\,\overline{\nabla\boldsymbol{v}}=v''(y)\,\boldsymbol{n}\,\overline{\boldsymbol{n}\,\hat{\boldsymbol{v}}} \tag{A.42}$$

where the prime denotes differentiation with respect to y. If no-slip boundary conditions are used and without magnetic field, $\eta\,v''(y)$ is equal to the pressure difference between both ends of the tube divided by its length.

Notice that $\delta\boldsymbol{f}=\nabla\,\delta p$ as given by (A.41) has components parallel to $\hat{\boldsymbol{v}}$, $\boldsymbol{n}$, and $\hat{\boldsymbol{v}}\times\boldsymbol{n}$. So far, the "longitudinal" and the "transverse" magnetic-field induced pressure gradients $\hat{\boldsymbol{v}}\cdot\delta\boldsymbol{f}$ and $(\hat{\boldsymbol{v}}\times\boldsymbol{n})\cdot\delta\boldsymbol{f}$ have been measured. HULSMAN and KNAAP [28] discussed recently which linear combination of viscosity coefficients is measured for a given direction of the magnetic field.

i) Longitudinal Pressure Gradient

The longitudinal component of $\nabla\,\delta p$ can be determined in a Wheatstone-bridge arrangement as used by ENGELHARDT and SACK [29], BEENAKKER et

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[28] H. HULSMAN and H. F. P. KNAAP, Physica **50**, 565 [1970].
[29] H. ENGELHARDT and H. SACK, Phys. Z. **33**, 724 [1932].

al. [1], KORVING et al. [20], and HULSMAN and BURGMANS [26] (see also Ref. [5]). With the help of the formula given in this Appendix the following expression for $\check{\boldsymbol{v}}\cdot\delta\mathbf{f}$ is inferred from (A.41):

$$\hat{\boldsymbol{v}}\cdot\delta\mathbf{f}\equiv\hat{\boldsymbol{v}}\cdot\nabla\delta p\;=\eta\,v''\{3\,\varepsilon_0\,h_y^2\,h_z^2+\varepsilon_1\,(h_y^2+h_z^2-4\,h_y^2\,h_z^2)\;+\varepsilon_2(1-h_y^2-h_z^2\;+h_y^2\,h_z^2)\}\,, \tag{A.43}$$

where
$$h_x=(\boldsymbol{n}\times\hat{\boldsymbol{v}})\cdot\boldsymbol{h}\,,\;h_y=\boldsymbol{n}\cdot\boldsymbol{h}\,,\;h_z=\hat{\boldsymbol{v}}\cdot\boldsymbol{h};\;h_x^2+h_y^2+h_z^2=1\,.$$

For special directions of $\boldsymbol{h}$ (specifying the direction of the magnetic field) Eq. (A.43) reduces to

$$\begin{aligned}
\hat{\boldsymbol{v}}\cdot\delta\mathbf{f}&=\eta\,v''\,\varepsilon_2 &&\text{for}\quad \boldsymbol{h}\perp\boldsymbol{v}\quad\text{and}\quad\boldsymbol{h}\perp\boldsymbol{n}\,,\\
\hat{\boldsymbol{v}}\cdot\delta\mathbf{f}&=\eta\,v''\,\varepsilon_1 &&\text{for either}\quad \boldsymbol{h}\parallel\boldsymbol{n}\quad\text{or}\quad\boldsymbol{h}\parallel\hat{\boldsymbol{v}}\,,\\
\hat{\boldsymbol{v}}\cdot\delta\mathbf{f}&=\eta\,v''\,\tfrac{1}{4}\quad(3\,\varepsilon_0+\varepsilon_2) &&\text{for}\quad \boldsymbol{h}\parallel(\boldsymbol{n}\pm\hat{\boldsymbol{v}})\,.
\end{aligned} \tag{A.44}$$

Hence such measurements allow the determination of the coefficients ε_0, ε_1, and ε_2.

ii) Transverse Pressure Gradient

The transverse component of $\nabla\,\delta p$ which gives rise to a "transverse" magnetic-field induced pressure difference is given by

$$\begin{aligned}(\hat{\boldsymbol{n}}\times\boldsymbol{v})\cdot\delta\mathbf{f}\;=(\hat{\boldsymbol{n}}\times\boldsymbol{v})\cdot\nabla\,\delta p\;&=\eta\,v''\,\{h_x\,h_z[3\,\varepsilon_0\,h_y^2\;+\varepsilon_1(1-4\,h_y^2)\;+\varepsilon_2(h_y^2-1)]\\
&\quad-h_y[\varepsilon_1^{\text{trans}}(1-2\,h_y^2)\;+\varepsilon_2^{\text{trans}}(h_y^2-1)]\}\,.\end{aligned} \tag{A.45}$$

The terms containing ε_0, ε_1, ε_2 give no contribution to the transverse pressure gradient if $\boldsymbol{h}$ is perpendicular to $\hat{\boldsymbol{v}}$ or perpendicular to $\boldsymbol{n}\times\hat{\boldsymbol{v}}$. In particular, Eq. (A.45) reduces to

$$(\boldsymbol{n}\times\hat{\boldsymbol{v}})\cdot\delta\mathbf{f}=\eta\,v''\,\varepsilon_1^{\text{trans}}\qquad\text{for }\boldsymbol{h}\parallel\boldsymbol{n}$$

and to

$$(\boldsymbol{n}\times\hat{\boldsymbol{v}})\cdot\delta\mathbf{f}=\eta\,v''\,\frac{1}{2\sqrt{2}}\,\varepsilon_2^{\text{trans}}\qquad\text{for }h_y=\frac{1}{\sqrt{2}}\,(\boldsymbol{h}\parallel\boldsymbol{n}\pm\hat{\boldsymbol{v}}\text{ or }\;\boldsymbol{h}\parallel\boldsymbol{n}\pm\widehat{(\boldsymbol{n}\times\boldsymbol{v})})\,. \tag{A.46}$$

Measurements of this type have been performed by KORVING et al. [20], HULSMAN and BURGMANS [26], and by KIKOIN et al. [30].

If $\boldsymbol{h}$ is perpendicular to $\boldsymbol{n}$, Eq. (A.45) yields

$$(\boldsymbol{n}\times\hat{\boldsymbol{v}})\cdot\delta\mathbf{f}=\eta\,v''\,h_x\,h_z(\varepsilon_1-\varepsilon_2)\,. \tag{A.47}$$

The pressure difference due to (A.47) has been measured by KIKOIN et al. [30].

Throughout this paper, the cross-effect between the traceless and the scalar parts of the friction pressure tensor which may exist in the presence of a magnetic field has been disregarded. It seems worth mentioning that the longitudinal and the transverse components $\hat{\boldsymbol{v}}\cdot\nabla\,\delta p$ and $(\boldsymbol{n}\times\hat{\boldsymbol{v}})\cdot\nabla\,\delta p$ are not affected by the existence of such an effect. This, however, is not true for the normal component $\boldsymbol{n}\cdot\nabla\,\delta p$ of the magnetic-field induced pressure gradient, as will be explained immediately.

iii) Normal Pressure Gradient

After (A.41) the component of $\nabla\,\delta p$ parallel to $\boldsymbol{n}$ is given by

$$\boldsymbol{n}\cdot\nabla\,\delta p=\eta\,v''\,\{h_y\,h_z[\varepsilon_0(3\,h_y^2-1)\;+\varepsilon_1(2-4\,h_y^2)\;+\varepsilon_2(h_y^2-1)]\;\;-h_x[2\,\varepsilon_1^{\text{trans}}\,h_y^2\;+\varepsilon_2^{\text{trans}}(1-h_y^2)]\}\,. \tag{A.48}$$

Here a remark on the fore-mentioned cross-effect is in order. The constitutive law for the scalar part p' of the friction pressure tensor is [25]

$$p'=-3\,\xi\,\overline{\boldsymbol{h}\boldsymbol{h}}:\overline{\nabla\boldsymbol{v}}-\eta_{\text{V}}\,\nabla\cdot\boldsymbol{v} \tag{A.49}$$

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[30] I. K. KIKOIN, K. I. BALASHOV, S. D. LASAREV, and R. E. NEISHTADT, Phys. Letters **24 A**, 165 [1967]; **26 A**, 650 [1968].

where η_{V} is the volume (bulk) viscosity and ξ is a cross coefficient which vanishes for $\boldsymbol{H}=0$. Hence, for the non-divergent flow between two flat plates the cross-effect gives rise to an additional static pressure gradient

$$\nabla p_{\mathrm{add}} = 3\,\xi\,v''\,h_y\,h_z\,\boldsymbol{n}\,, \tag{A.50}$$

which has only a component parallel to $\boldsymbol{n}$.

The total static pressure gradient in $\boldsymbol{n}$-direction induced by a magnetic field is given by the sum of (A.48) and (A.50); e. g. if $\boldsymbol{h}$ is perpendicular to $\boldsymbol{n}\times\boldsymbol{v}$ one has

$$\boldsymbol{n}\cdot\nabla\,\delta p_{\mathrm{tot}} = \frac{2\sqrt{2}}{9}\,\eta\,v''(\varepsilon_1-\varepsilon_2) + \sqrt{2}\,\xi\,v'' \quad \text{for} \quad h_y=\frac{1}{\sqrt{3}}\,,\quad h_z=\sqrt{\frac{2}{3}}$$

and

$$\boldsymbol{n}\cdot\nabla\,\delta p_{\mathrm{tot}} = \tfrac{1}{4}\,\eta\,v''(\varepsilon_0-\varepsilon_2) + \tfrac{3}{2}\,\xi\,v'' \quad \text{for} \quad h_y=h_z=\frac{1}{\sqrt{2}}\,.$$

For known ε_0, ε_1, ε_2 such a measurement could be used to obtain experimental values for ξ. Subject to the assumption that the tensor polarization is the only type of alignment set up by a viscous flow a theoretical analysis yields $\varepsilon_0=0$ and $\xi=0$. If other types of alignment are present $\varepsilon_0 \neq 0$ and $\xi \neq 0$ is obtained[9]. However, ε_0 and ξ/η can be expected to be small compared with ε_1 and ε_2.

Zur Verbreiterung der Wasserstoff-Linie Ly-α durch starke Elektronenstöße

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(Z. Naturforsch. **26 a**, 1071—1087 [1971]; eingegangen am 16. März 1971)

On Stark Broadening of the Hydrogen Line Ly-α by Strong Electron Collisions

A study of how strong electron collisions will influence the profile of the line core is made within the framework of the impact theory. To this end the time-dependent Schrödinger-equation describing the change of state of the atom during the passage of one perturbing electron is solved numerically down to impact parameters equal to the de Broglie-wavelength $\lambda\!\!\!^{-}$. In this range the classical description of the perturbing electrons is assumed to be still valid. In the dipole-only case the solution can be expressed exactly by elementary functions. The line profile clearly shows how important it is to consider the strong collisions. All the different ways of doing this lead to nearly the same results.

1. Einleitung

Unabhängig voneinander haben BARANGER[1] und GRIEM und KOLB[2] eine Theorie entwickelt, um die Verbreiterung (und Verschiebung[3]) der Spektrallinien eines Atoms in einem Plasma zu erklären. Drei grundlegende Annahmen kennzeichnen diese Theorie:

a) Die Elektronen und Ionen des Plasmas dürfen als Teilchen im Sinne der klassischen Mechanik behandelt werden.

b) Die Wechselwirkung der Elektronen mit dem Atom läßt sich in eine Folge von Zwei-Teilchen-Stößen Atom – Elektron zeitlich auflösen, die Übergänge zwischen den quantenmechanischen Zuständen des Atoms bewirken (verallgemeinerte Stoßdämpfungstheorie).

c) Demgegenüber kommt es auf die Zeitabhängigkeit des von den Ionen am Ort des Atoms erzeugten elektrischen Mikrofeldes nicht an, die einzelnen Komponenten, in die die Spektrallinie infolge des Stark-Effektes aufspaltet und die gemäß b) eine Elektronenstoßverbreiterung aufweisen, sind nach Maßgabe ihrer Intensität und der Mikrofeldverteilungsfunktion zu überlagern (quasistatische Theorie).

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* Sonderdruckanforderungen an Dr. H. PFENNIG, Max-Planck-Institut für Physik und Astrophysik, *D-8000 München 23*, Föhringer Ring 6.

[1] M. BARANGER, Phys. Rev. **111**, 494 [1958].

[2] H. R. GRIEM u. A. C. KOLB, Phys. Rev. **111**, 514 [1958].

[3] Mit der Verbreiterung soll im folgenden zugleich auch immer die Verschiebung gemeint sein, wenn nicht etwas anderes ausdrücklich bemerkt ist.